# Tripulse: A System for Determining Orientation and Attitude of a Satellite Borne Active Phased Array


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**Abstract**

Tripulse is a novel system that is designed to accurately estimate the orientation of a satellite borne phased array relative to one or more earth stations. It has conceptual similarities to amplitude comparison monopulse systems used in tracking radars. Tripulse differs from monopulse in that it is applicable primarily to transmitting active phased arrays aboard communication satellites. The signaling structure consists of a minimum of three separate time multiplexed coherent pulses. A statistical performance analysis and a description of the methodology for converting multiple ground station orientation data to attitude estimates are presented.


## 1. Introduction

A communication satellite system utilizing active phased array antenna technology can require a higher degree of pointing accuracy than that which can be obtained using Earth-Moon and Sun sensors. Accurate attitude stabilization is imperative for the delivery of maximum signal power to localized areas and for limiting interference in frequency reuse applications. Earth-Moon-Sun sensor systems typically provide an attitude accuracy of $\sim 0.1$ degrees [1]. More sophisticated systems that include star sensors can improve the attitude accuracy to $\sim 0.05$ degrees. The component cost and added weight are significant factors that negatively impact their desirability.

This work describes a novel method called *Tripulse* which provides an accurate estimate of the orientation of a satellite borne phased array. The system control computer is programmed to provide highly accurate attitude estimates of the satellite phased array using a combination of orbital data coupled with Tripulse orientation data obtained from two or more participating ground stations. Attitude in this context refers to the orientation of the array coordinate system relative to an inertial frame of reference [1].

The Tripulse method relies upon the existence of a *reasonably good* initial orientation estimate that can be obtained using conventional attitude measurement systems [1]. Using


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the initial estimate as the starting point, the Tripulse system has the capability of increasing the accuracy by several orders of magnitude. As a result, the accuracy requirement of the optical sensors in a system incorporating Tripulse is less stringent, existing beamforming hardware and control software largely provide the needed functionality for implementation.

The Tripulse system is conceptually similar to amplitude comparison monopulse systems used in tracking radars, see for example [2,10]. The Tripulse system differs from conventional monopulse in that it is applicable to receive *or* transmit phased arrays. However, the single pulse concept, for which monopulse is named, is no longer feasible. In Tripulse, a minimum of three coherent pulses must be used. The coherent detection system must be designed to compensate for doppler shifts due to the motion and phase noise effects from the unsynchronized clocks on the satellite and the ground station. In order to assure that coherence is maintained, the pulses have to be accompanied by one or more reference signals. A number of reference signal architectures are possible. Two examples are as follows:

1. Let the tripulse signals be tones at frequency $f_t$. Transmit two reference tone signals at frequencies $f_{r1}, f_{r2}$, where, $f_{r1} + f_{r2} = 2f_t$. By appropriately mixing and filtering the resulting signals in the coherent detection system, phase migration effects can be canceled out exactly [3].

2. The composite signal pulses can be segmented and interleaved for this approach. Here temporal phase migration as well as signal integrity (in the event of unstable atmospheric conditions) can be ascertained by implementing a Kalman amplitude and phase tracker on the multiple samples of the sum beam.

This paper is organized as follows. Section 2 serves as a review of the pertinent aspects of monopulse tracking that are common to the Tripulse system. A discussion of the requirements unique to the Tripulse system is included in Section 3. Section 4 presents performance analyses of the Tripulse system, including the effects of noise, beamforming quantization errors, and failed array beamforming components. Section 5 will present a method to obtain the attitude estimate of the satellite array based upon Tripulse orientation data. Concluding remarks are offered in Section 6.

## 2. Background: Amplitude Comparison Monopulse/Tripulse

Amplitude monopulse tracking works in the following way. A radar antenna scans the horizon transmitting a pulsed spot-beam pattern that illuminates potential targets. The scan is accomplished either by mechanical rotation of the array or by electronic steering. The position of the beam at the time the echo signal is received establishes the initial estimate of the target's position. At the radar receiver, each individual echo pulse is divided into three separate coherent signals. Each signal corresponds to a different setting of the elemental phases within the receive beamformer. The three receiving beam patterns are a *sum* beam characterized by a uniform phase plane over the array and two other *delta* beams, $\Delta_x$ and $\Delta_y$. In the sum beam, the normal to the uniform phase plane is directed to the target of interest. The delta beams are obtained by changing the phases of the receive array elements by $\pi$ on either side of the $x, y$ symmetry axes. These signals are then processed to determine the orientation of the target relative to the array.

Radar monopulse tracking can be degraded by undesirable sources of energy in the sidelobes of the receiving beams. These sources can be extraneous scatterers such as decoys or RF sources such as radar jammers. The undesirable effects can be mitigated to varying degrees by implementing amplitude tapers on the receive beams to suppress the diffraction sidelobes. Aperture tapering can be used in Tripulse, but it should be used only if there is a potential jamming problem. In situations where there are no extraneous sources, aperture shading is undesirable because it reduces the aperture efficiency and, commensurately, the SNR of the detection process.

For simplicity consider a rectangular phased array antenna with an even number of elements in each row and column. Our array model will consist of individual active elements that can be assigned a distinct amplitude and phase. This model provides a simplified mathematical formulation from which closed form analytical results can be obtained. However, Tripulse is not restricted to systems that conform to this array model. As with conventional monopulse, the only requirement is that the shapes of the component beams be described analytically or by accurate measured pattern data.

Let the $x, y$ axes of the array coordinate system lie in the plane of the array passing through the center of symmetry and directed along the elemental rows and columns. The $z$ axis is perpendicular to plane of the array. The array is composed of $N_{\rm tot} = N_x \times N_y$ elements. For a right-handed coordinate system place the $(1, 1)$ element in the upper left hand corner. Here $m = 1, 2, \ldots, N_x$, and $n = 1, 2, \ldots, N_y$. The array is now divided into four quadrants,

$$
\begin{aligned}
Q_1 &\rightarrow m \in (1, N_x/2), \quad n \in (1, N_y/2), \\
Q_2 &\rightarrow m \in (N_x/2+1, N_x), \quad n \in (1, N_y/2), \\
Q_3 &\rightarrow m \in (N_x/2+1, N_x), \quad n \in (N_y/2+1, N_y), \\
Q_4 &\rightarrow m \in (1, N_x/2), \quad n \in (N_y/2+1, N_y).
\end{aligned} \tag{1}
$$

The angular coordinates of the target or, equivalently, the ground receiving system for the phased array satellite, are characterized in spherical polar coordinates by $(R, \theta, \phi)$. Here $\theta$ and $\phi$ are respectively the polar and azimuthal angles. The center of the array is designated as the origin of the coordinate system. The projections of the unit vector directed towards the target (ground station) from the array center are given by

$$\begin{aligned} T_x &\stackrel{\text{def}}{=} \sin\theta\,\cos\phi, \\ T_y &\stackrel{\text{def}}{=} \sin\theta\,\sin\phi, \\ T_z &\stackrel{\text{def}}{=} \cos\theta. \end{aligned} \tag{2}$$

The initial estimates of the angular coordinates of the target, $(\theta_0, \phi_0)$, are represented by initial cartesian projections, $T_{x0}, T_{y0}, T_{z0}$.

Assume that the target at $\vec{R}$ is in the far-field of the transmitting/receiving array. The coherent signals received at the individual array elements for radar monopulse, or equivalently at the ground station for tripulse, are of the form

$$s(m,n) \propto \frac{e^{jkR}}{R} \exp\Big[\frac{-jk\vec{r}_{mn}\cdot\vec{R}}{R}\Big]. \tag{3}$$

The projections of $\vec{R}$ onto the vector coordinates of the array elements, $\vec{r}_{mn}$, are given by

$$\frac{\vec{r}_{mn}\cdot\vec{R}}{R} = c(m)d_xT_x + c(n)d_yT_y. \tag{4}$$

Here $k$ is the wave-number, $2\pi/\lambda$, $d_x, d_y$ are the array element spacings along the x,y directions, and

$$\begin{aligned} c(m) &= m - (N_x+1)/2; \text{ for } m = 1, 2, \ldots, N_x, \\ c(n) &= n - (N_y+1)/2; \text{ for } n = 1, 2, \ldots, N_y. \end{aligned} \tag{5}$$

The notation can be further simplified by grouping the proportionality constant representing the combined effects of propagation and random phase offsets that are common to all the elements into one complex coefficient $K$.

Let $\{S_{Q_i}\}$ represent steered beams corresponding to the sum of all the individual coherent received (transmitted) elemental signals in each of the quadrants. Defining $X \stackrel{\text{def}}{=} kd_x(T_x - T_{x0})$, and $Y \stackrel{\text{def}}{=} kd_y(T_y - T_{y0})$, the signals $\{S_{Q_i}\}$ are:

$$\begin{aligned} S_{Q_1} &= K\sum_{m=1}^{N_x/2} e^{-jc(m)X} \sum_{n=1}^{N_y/2} e^{-jc(n)Y}, \\ S_{Q_2} &= K\sum_{m=N_x/2+1}^{N_x} e^{-jc(m)X} \sum_{n=1}^{N_y/2} e^{-jc(n)Y}, \\ S_{Q_3} &= K\sum_{m=N_x/2+1}^{N_x} e^{-jc(m)X} \sum_{n=N_y/2+1}^{N_y} e^{-jc(n)Y}, \\ S_{Q_4} &= K\sum_{m=1}^{N_x/2} e^{-jc(m)X} \sum_{n=N_y/2+1}^{N_y} e^{-jc(n)Y}. \end{aligned} \tag{6}$$

The sum and difference of the steered received beams used in the monopulse method are defined by

$$\begin{aligned} S_\Sigma &= S_{Q_1} + S_{Q_2} + S_{Q_3} + S_{Q_4}, \\ S_{\Delta x} &= S_{Q_1} - S_{Q_2} - S_{Q_3} + S_{Q_4}, \\ S_{\Delta y} &= S_{Q_1} + S_{Q_2} - S_{Q_3} - S_{Q_4}. \end{aligned} \tag{7}$$

The analytical expressions for these signals are obtained from computing the sums in Equation 6,

$$\begin{aligned} S_\Sigma &= K \frac{\sin(XN_x/2)}{\sin(X/2)} \frac{\sin(YN_y/2)}{\sin(Y/2)}, \\ S_{\Delta_x} &= j2K \frac{\sin^2(XN_x/4)}{\sin(X/2)} \frac{\sin(YN_y/2)}{\sin(Y/2)}, \\ S_{\Delta y} &= j2K \frac{\sin(XN_x/2)}{\sin(X/2)} \frac{\sin^2(YN_y/4)}{\sin(Y/2)}. \end{aligned} \tag{8}$$

From these results we see that the sum and delta signals differ in phase by $\pi/2$. The quadrature relationship is a result of the even and odd symmetry of the sum and delta beam weights, respectively. This is still generally true whenever tapering is applied to the aperture.

The imaginary parts of the ratio of the delta to the sum beams are usually referred to as the *Monopulse ratios.* Measurements of these terms are used to establish the error in the initial estimate of the orientation angles,

$$\begin{aligned} \mathcal{R}_x &\stackrel{\text{def}}{=} \Im m\left[\frac{S_{\Delta x}}{S_\Sigma}\right] = 2\frac{\sin^2(XN_x/4)}{\sin(XN_x/2)} = \tan(XN_x/4), \\ \mathcal{R}_y &\stackrel{\text{def}}{=} \Im m\left[\frac{S_{\Delta y}}{S_\Sigma}\right] = \tan(YN_y/4). \end{aligned} \tag{9}$$

The process that is used to estimate the orientation errors is as follows:

1. Measure the ratios, $\hat{\mathcal{R}}_x, \hat{\mathcal{R}}_y$. These are the noisy measurements, hence they are estimates of $\mathcal{R}_x, \mathcal{R}_y$.

2. Given the measured values, solve the transcendental equations in (9) for $\hat{X}N_x/4$ and $\hat{Y}N_y/4$. From these solutions the error in the initial estimates are

$$\hat{\epsilon}_x \stackrel{\text{def}}{=} \hat{T}_x - T_{x0}, \quad \hat{\epsilon}_y \stackrel{\text{def}}{=} \hat{T}_y - T_{y0}.$$

The estimated polar and azimuthal angles can be computed using

$$\hat{\theta} = \sin^{-1}\sqrt{\hat{T}_x^2 + \hat{T}_y^2}; \quad \hat{\phi} = \tan^{-1}\left(\frac{\hat{T}_y}{\hat{T}_x}\right). \tag{10}$$

The monopulse/tripulse methods are fundamental prescriptions to find the steering angle of a sum beam that maximizes its received power. This is most significant as it is well known from spectral estimation theory that the maximum likelihood estimate for a single sinusoid in complex Gaussian white noise corresponds to the maximum of the periodogram [4].

## 3. Tripulse System

The Tripulse methodology of using sum and difference beams to obtain the orientation angle estimates is similar to that of the monopulse method discussed in the previous section. However, there are significant differences described as follows. A transmit-mode formulation of Tripulse is implied; Figure 1A shows the high-level block diagram.

1. To determine the orientation of a transmitting active phased array antenna, the sum and difference beams in Tripulse are formed on transmit rather than receive. Moreover the tripulse system must transmit three time multiplexed coherent pulses corresponding to the sum and two delta beams. The transmitting antenna is electronically steered to the initial position estimate (or nominal location) of the ground station. The initial orientation angle estimates are obtained either using separate Earth-Moon-Sun sensors, or the estimates from a previous Tripulse estimation sequence.

2. The receiving ground based antenna can either be a phased array or a dish antenna. The receiving antenna is steered to maximize the power from the satellite borne transmitting phased array; the receive pattern does not change over the duration of the Tripulse process.

3. Tripulse estimation requires that the full complex nature of the three beams be maintained. In order to coherently extract the complex magnitude of the Tripulse signals, one or more additional reference signals may be transmitted over a static channel by the satellite. The tripulse tone and reference signals are generated from a common reference. One such system described in [3,11] effectively compensates for Doppler phase shifts due to (1) satellite motion and (2) independent phase noise effects arising from non-synchronized clocks between the satellite and ground station. The auxiliary signals are used to create an amplitude and phase reference. The two delta beam signals, however small in magnitude, are then measured in a coherent fashion relative to that of the sum beam signal.

The terrestrial receiver features a coherent detection system exploiting the availability of the auxiliary reference tones. Following coherent detection, numerical processing provides estimates of the orientation angles. The processing of the Tripulse signals are identical in functional form to the corresponding monopulse signals given in Equation 8. The corresponding $\Delta/\Sigma$ ratios are given by Equation 9. Orientation estimates associated with multiple displaced stations are combined to provide the three dimensional attitude error as described in Section 5.

For the receive panel implementation shown in Figure 1B, the signal and reference tones are transmitted from the ground. Here also, the transmitting ground based antenna can either be a phased array or a dish antenna. The transmitting antenna is steered to maximize the signal power at the satellite borne receive array. The sum and delta beams are formed sequentially in the receive beamformer.

A system used to determine the orientation angles for a receiving satellite borne phased array can be made essentially identical to radar monopulse if, and only if, the onboard microwave hardware has the capability of splitting the single received pulse into the required three monopulse beams. Then the only difference between classical radar monopulse and Tripulse is that the receiving phased array is not used to illuminate the target, but rather the target itself generates the signal. In most cases, the satellite borne phased array receiving communication antenna will not possess the requisite hardware to simultaneously split the beam into the required $\Sigma$ and $\Delta$ components.

As with transmit Tripulse, the initial estimates are obtained either using separate Earth-Moon-Sun sensors, or the results from a previous Tripulse trial. The information processing can either be performed by an onboard satellite processor or on the ground via the transfer of raw data on the Telemetry, Tracking and Command (TT&C) communication link.

## 4. Tripulse performance analysis

In this section, the theoretical performance of the Tripulse direction estimation technique is derived. Particular analyses of interest are the estimation accuracy resulting from receiver noise, beamforming quantization, and failed elements. Appropriate to the commercial satellite application, the process time required to achieve a specified estimation accuracy is included. Due to the similarities with conventional monopulse, much appears in the literature that is applicable here, *e.g.* [10].

### 4.1 Statistical Analysis

In Section 2, we discussed how the estimates of the orientation errors, $\hat{\epsilon}_x$, $\hat{\epsilon}_y$, are obtained by solving the transcendental equations generated by the functional form of the $\Delta/\Sigma$ ratios given by Equation 9. For small errors in the initial orientation estimates, the functions can be approximated by their leading order Taylor series expansion terms,

$$\mathcal{R}_x = \frac{XN_x}{4} + \frac{1}{3}\Big(\frac{XN_x}{4}\Big)^3 + \mathcal{O}\Big(\frac{XN_x}{4}\Big)^5,$$

$$\mathcal{R}_y = \frac{YN_y}{4} + \frac{1}{3}\Big(\frac{YN_y}{4}\Big)^3 + \mathcal{O}\Big(\frac{YN_y}{4}\Big)^5. \tag{11}$$

Consider a typical Ku band system with $N_x = N_y = 16$, $d_x = d_y = 3\lambda$. For Earth-Moon-Sun attitude sensors, the error in the initial estimates, $|T_x - T_{x0}|$ and $|T_y - T_{y0}|$ are $\sim 0.1\pi/180$. Hence the nonlinear cubic terms in the expansions illustrated in Equation 11 are only $\sim 2\%$ of the corresponding linear terms. In this regime, we can analytically calculate the noise statistics of the projection errors in terms of the corresponding noise statistics of the $\Delta_x/\Sigma$ ratios,

$$\begin{aligned} E\{\epsilon_x\} &\cong \frac{4}{kd_xN_x} E\Big\{\mathcal{R}_x\Big\}, \\ \mathbf{var}\{\epsilon_x\} &\cong \Big(\frac{4}{kd_xN_x}\Big)^2 \mathbf{var}\{\mathcal{R}_x\}. \end{aligned} \tag{12}$$

Proper operation of the Tripulse system depends upon the coherent detection of the component signals. The noise at the receiver prior to demodulation can be modeled as zero-mean complex additive white Gaussian noise (AWGN) with power spectral density $\mathcal{N}_0$. As Silverstein has shown in his performance analyses of demodulated signals [5], the noise model must be modified because the demodulation process both band limits and energy limits the noise. Moreover, he showed that indiscriminate use of AWGN in performance analyses of coherent detection systems after demodulation can lead to analytical divergences.

In Silverstein's paper the difficulties with the AWGN were eliminated by the use of a modified model referred to as additive truncated Gaussian noise (ATGN). ATGN is represented by independent amplitude and phase random variables with probability density functions (pdf's) characterized respectively by a truncated Rayleigh amplitude distribution and a uniform phase distribution.

Assume the demodulated signals have been low-passed filtered to bandwidth $W$. The signals are subsequently sampled at $W = 1/T$, where $T$ is the sampling period. The ATGN samples $\{n_i\}$ are statistically independent. In the moderate SNR regime the statistics of the complex ATGN noise can be approximated by the statistics of complex AWGN,

$$E\{n_i^m\} = 0; \quad E\{(n_i n_j^*)^m\} = m!\sigma^{2m}\delta_{ij}. \tag{13}$$

The variance of the noise samples has an implicit bandwidth dependence, $\sigma^2 \equiv \mathcal{N}_0 W$. The expressions for the mean and variance of $\hat{\mathcal{R}}_x$ are given by,

$$E\Big\{\hat{\mathcal{R}}_x\Big\} = E\Big\{\Im m\Big[\frac{S_{\Delta_x}+n_2}{S_\Sigma+n_1}\Big]\Big\} = E\Big\{\Im m\Big[(S_{\Delta_x}+n_2)\Big(1/S_\Sigma - n_1/S_\Sigma^2 + ...\Big)\Big]\Big\} \tag{14}$$

$$= \Im m\Big[\frac{S_{\Delta x}}{S_\Sigma}\Big] \tag{15}$$

$$\mathbf{var}\Big(\hat{\mathcal{R}}_x\Big) = E\Big\{\Big|\Im m\Big[\frac{S_{\Delta_x}+n_2}{S_\Sigma+n_1}\Big]\Big|^2\Big\} - \Big|E\Big\{\Im m\Big[\frac{S_{\Delta_x}}{S_\Sigma}\Big]\Big\}\Big|^2. \tag{16}$$

Similar terms prevail for the mean and variance of $\hat{\mathcal{R}}_y$. Defining $\gamma_x \equiv |S_{\Delta x}/S_\Sigma|$, the expansion for the variance can be rearranged as

$$\mathbf{var}\Big(\hat{\mathcal{R}}_x\Big) = \Big(1+\gamma_x^2\Big)\frac{\sigma^2}{2|S_\Sigma|^2} + \Big(1-\gamma_x^2\Big)\frac{\sigma^4}{2|S_\Sigma|^4} + \mathcal{O}\Big(\frac{\sigma^6}{|S_\Sigma|^6}\Big). \tag{17}$$

In the intended application, $|S_\Sigma|^2 >> \sigma^2$, and $\gamma_x^2 << 1$. From Equations 12 and 16,

$$\mathbf{var}\{\epsilon_x\} \cong \Big(\frac{\lambda}{\pi d_x}\Big)^2 \frac{2}{N_x^2 N_{\text{tot}}^2}\frac{\mathcal{N}_0 W}{|K|^2} \tag{18}$$

Here $|K|^2/(\mathcal{N}_0 W)$ is the link budget value of the receiver SNR of the signal transmitted from *a single element* of the phased array. Recall that $N_{\text{tot}}$ is the total number of array elements.

We want to relate the noise statistics of the polar angle estimates $\hat{\theta}$ to those of the projection errors, $\hat{\epsilon}_x, \hat{\epsilon}_y$. A Taylor series expansion shows that the expected value of $\hat{\theta} = \sin^{-1}\left(\sqrt{T_x^2+T_y^2}\right)$ is $\theta_0$ to first order, where $\sin^2\theta_0 \equiv T_{x0}^2 + T_{y0}^2$. Let $\sigma_T^2$ represent the variance of the estimates $\hat{T}_x$ and $\hat{T}_y$. Therefore,

$$E\{\sin^2\hat{\theta}\} = E\{\hat{T}_x^2 + \hat{T}_y^2\} = T_{x0}^2 + T_{y0}^2 + 2\sigma_T^2. \tag{19}$$

For small errors,

$$E\{\sin^2\theta\} \cong \sin^2\theta_0 + \sigma_\theta^2(\cos^2\theta_0 - \sin^2\theta_0), \tag{20}$$

so that

$$\mathbf{var}\left(\hat{\theta}\right) \sim 2\mathbf{var}(\epsilon_x)/cos2\theta_0. \tag{21}$$

Using a similar procedure to that given above, we can derive the relation between the variance of the projections and the azimuthal angle, $\sigma_T^2 \cong \sin^2\theta\sigma_\phi^2$.

In Figure 2 we show the simulation results for the standard deviation of the polar angle estimates for different single element SNR levels as a function of the number of elements in the array. In our simulations, the initial angular estimate was taken at a fixed error of 0.1 degrees. This error value is three times the standard deviation of the error estimates derived from conventional Earth-Moon-Sun attitude sensors. The simulation results illustrated in Figure 2 are in accord with the theoretical results of Equation 21.

### 4.2 Tripulse measurement time estimates

Application of the Tripulse method to communication satellites requires that a transmit and/or receive beamformer be taken offline for the formation of the monopulse beams. As a result, the viability of the method depends upon how long it takes to obtain an accurate estimate of the orientation angles. As we show below, the required measurement time for large order arrays can be very short. This is due to direct relationship between the estimation SNR and the gain of the sum beam.

From Equation 18 with $T_{\text{int}} \equiv 1/W$, the integration time necessary to satisfy a specification on the orientation error, $\delta_{\text{spec}} \equiv \mathbf{std}(\epsilon_x) \equiv \mathbf{std}(\epsilon_y)$, is

$$T_{\text{int}} \geq \frac{1}{\delta_{\text{spec}}^2} \frac{2}{N_x^2 N_{\text{tot}}^2} \left( \frac{\lambda}{\pi d_x} \right)^2 \frac{\mathcal{N}_0}{|K|^2}. \tag{22}$$

Consider a Ku band satellite system in a geostationary orbit. Assume a square array composed of $N_{\text{tot}} = 256$ elements, where the spacing is appropriate for whole-earth coverage ($d_x = d_y = 3\lambda$ separation). Let the single-element downlink budget value be $|K|^2/\mathcal{N}_0 \sim$ 37dBHz, and an orientation-projection specification $\delta_{\text{spec}} = 5.8 \times 10^{-5}$. This corresponds to a *3 sigma* (estimate within 3 standard deviations of correct value) angular specification of .01 degrees. Substituting these values into Equation 22, we calculate a required measurement time of $\sim 80\mu$sec.

In some cases the interference is dominated by intermodulation signals rather than receiver noise. The intermodulation spectrum is often modelled as an equivalent white noise process, filtered to the receiver bandwidth. For this case, the analysis proceeds the same as above, where the noise spectral density $\mathcal{N}_0$ is replaced by the effective intermodulation spectral density, $\mathcal{I}_0$. Here again, 37 dBHz is a representative number for the downlink budget, and the required integration time will still be $\sim 80\mu$sec.

### 4.3 Sensitivity to quantization errors

Figure 3 illustrates the sensitivity of the Tripulse estimate to random phase quantization errors in the beamformer phase states. The *Monte Carlo* simulations were performed using phase errors that were uniformly distributed over a width equal to the quantization level associated with the highest bit state. For example, a five bit quantizer will have a uniformly distributed phase error of $\pm 1/2(2\pi/2^5)$. As we see, nominal Tripulse 3 sigma orientation error specifications of $\leq 0.01$ degrees are met for array orders exceeding $\sim 320$ elements for a five bit quantizer.

### 4.4 Sensitivity to faulty elements

Over the lifetime of a phased array satellite system, elemental degradation will occur. The question to answer is: how does elemental degradation effect the accuracy of the orientation estimate? As amplifier failure is the most common mechanism, we answer the above posed question by assessing the sensitivity of the Tripulse procedure to failed elements.

The results of our *Monte Carlo* simulations are illustrated in Figure 4. To generate these results, we used the 256 element array described in Section 4.1. We consider the case of zero receiver noise to isolate the effects of element failure. The error statistics were computed over 20,000 independent trials. The arrangement of "dead" elements were randomized for each trial. The results indicate that in the absence of receiver noise, 3 sigma specifications of $\sim 0.01$ degrees can be maintained for up to $\sim 15\%$ element failures. The decrease in the Tripulse SNR caused by failed elements can be readily compensated by increasing the measurement integration time as discussed in Section 4.2.

## 5. Attitude Estimation

### 5.1 Attitude determination

The goal of three-axis attitude estimation is to find the physical rotation the array coordinate system has undergone relative to some fixed reference coordinate system, assuming that the position of the array is known. The reference coordinate system may be an inertial frame or the roll, pitch and yaw axes used for satellites in Earth bound orbits. The direction to a remote receiver in the reference coordinate system is known *a priori*. The Tripulse technique measures the direction to a remote receiver relative to the array coordinate system. The attitude is determined by finding the transformation that registers these direction vectors.

If only one remote receiver is used and the array rotates about the imaginary line that connects the array to the remote receiver then the Tripulse measurements taken at that receiver will not change. Therefore, this mode of rotation is not observable by a lone remote receiver. In order to solve for the attitude of the phased array, direction measurements to at least two receivers are necessary, more can be used to increase accuracy and robustness.

Let $\mathbf{x}_n$ for $n = 1, \ldots, N_{\text{tot}}$ be a 3 dimensional vector representing the position of array element $n$ in the array coordinate system. The position of each element in the reference coordinate system is modeled by

$$\mathbf{x}'_n = \mathbf{R}^T \, \mathbf{x}_n, \tag{23}$$

where $\mathbf{R}$ is the unknown rotation matrix representing the array attitude. Since $\mathbf{R}$ is orthonormal, $\mathbf{R}^{-1} = \mathbf{R}^T$.

Let $l = 1, \ldots, L$ index the remote receivers. The direction to remote receiver $l$ is represented by the unit vector $\mathbf{t}'_l$ in the reference coordinate system and by $\mathbf{t}_l = (T_{xl},\, T_{yl},\, T_{zl})$ in the array coordinate system. The Tripulse procedure estimates $T_{xl}$ and $T_{yl}$ and we calculate $T_{zl} = \sqrt{1 - T_{xl}^2 - T_{yl}^2}$. The apparent rotation of the remote receivers is the inverse of the rotation of the array elements, so

$$\mathbf{t}_l = \mathbf{R} \, \mathbf{t}'_l. \tag{24}$$

Using this relation, Tripulse measurements are used to solve for the unknown attitude.

### 5.2 Least-squares estimate

The unknown rotation matrix or attitude can be specified by three parameters. A third order rotation matrix has nine real values but the orthonormality requirement reduces its number of free parameters to three. Tripulse measurements from two remote receivers provide four measurements in total. This overdetermines the array attitude by one parameter, so a least-squares solution is used.

A rotation matrix can be specified by an axis of rotation and the amount of rotation about that axis. Two parameters specify the axis and one specifies the amount of rotation. A rotation matrix can also be specified by roll, pitch and yaw angles. Below we will need to solve directly for rotation matrices, but these solutions can be converted to roll, pitch and yaw angles if desired.

Let $\hat{\mathbf{t}}_l = \mathbf{t}_l + \Delta\mathbf{t}_l$ be the noisy Tripulse generated estimates of $\mathbf{t}_l$, where $\Delta\mathbf{t}_l$ is additive noise. We wish to find the rotation matrix $\hat{\mathbf{R}}$ that minimizes the least-squares cost function

$$J = \sum_{l=1}^{L} w_l ||\hat{\mathbf{t}}_l - \hat{\mathbf{R}}\, \mathbf{t}'_l||^2. \tag{25}$$

The optional weight factors $w_l$ may be used to improve performance by emphasizing more reliable data. The rotation $\hat{\mathbf{R}}$ that minimizes $J$ best aligns, in the weighted least-squared sense, the known remote receiver directions in the reference coordinate system with the Tripulse generated receiver directions in the array coordinate system. Thus, the estimated rotation matrix is an estimate of the attitude of the array.

General solutions to this least-squares problem have been developed for two or more remote receivers [6,7,8]. However, for the special case of two remote receivers with equal weighting, a simple procedure for finding the least-squares rotation matrix is given here.

If $\mathbf{t}_1$ and $\mathbf{t}_2$ are separated by the same angle as $\mathbf{t}'_1$ and $\mathbf{t}'_2$ then a rotation matrix exists which rotates $\mathbf{t}_1$ and $\mathbf{t}_2$ exactly onto $\mathbf{t}'_1$ and $\mathbf{t}'_2$. These separation angles, however, will not be identical when the measurements are noisy. The squared error is minimized by a rotation matrix that transforms the measured vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, so the resultants, $\mathbf{R}^T\, \mathbf{t}_1$ and $\mathbf{R}^T\, \mathbf{t}_2$, satisfy the following two criteria: they are coplanar with the nominal direction vectors, $\mathbf{t}'_1$ and $\mathbf{t}'_2$; and their bisector is the same as that of the nominal direction vectors.

The simple procedure for finding this rotation matrix involves defining a special set of three orthonormal vectors for the measured direction vectors, and another corresponding set for the known direction vectors. One of the vectors bisects the pair of direction vectors and another is normal to the direction vectors, as defined below. These orthonormal vectors are defined below and shown in Figure 5.

The first orthonormal vector is the unit length bisector of the direction vectors

$$\mathbf{A}_1 = \frac{\mathbf{t}_1 + \mathbf{t}_2}{||\mathbf{t}_1 + \mathbf{t}_2||}.$$

The second is normal to the direction vectors, making it also normal to the bisector

$$\mathbf{A}_2 = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{||\mathbf{t}_1 \times \mathbf{t}_2||}.$$

The third vector completes a right handed set of axes

$$\mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2.$$

These three orthonormal vectors are assembled into a matrix as follows

$$\mathbf{A} = [\ \mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ ].$$

The matrix $\mathbf{A}'$ for the known direction vectors is found in a similar manner. This matrix can be computed in advance, though the computation savings is negligible.

Now, the least-squares rotation matrix is the rotation that registers the orthonormal vectors for the nominal pointing vectors with the orthonormal vectors for the measured pointing vectors,

$$\mathbf{R} = \mathbf{A}\ \mathbf{A}'^T.$$

Tripulse measurements from more than two remote receivers can be used to find a least-squares estimate of the attitude.

There are two fast methods from the computer vision field for solving this problem. One uses the Singular Value Decomposition (SVD) of a third order matrix [6,7]. The other uses quaternions [8]. The computation of an SVD is an iterative process. However, the third order SVD needed here takes an insignificant amount of computation. The method using quaternions gives a direct solution, but is much more algorithmically complex.

The SVD technique is described here. First, define the matrix

$$\mathbf{H} \stackrel{\text{def}}{=} \sum_{l=1}^{L} \mathbf{t}'_l\ \mathbf{t}_l^T.$$

Compute the SVD of matrix $\mathbf{H}$

$$\mathbf{H} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T.$$

By definition of the SVD, $\mathbf{U}$ and $\mathbf{V}$ are third order orthonormal matrices and $\mathbf{\Lambda}$ is a third order diagonal matrix of singular values. Compose a candidate solution

$$\mathbf{X} = \mathbf{V}\ \mathbf{U}^T.$$

This procedure finds an orthonormal matrix $\mathbf{X}$ that solves the least-squares problem. However, a third order orthonormal matrix can be either a rotation or a rotation with a reflection. Our final solution must be a pure rotation, but this is easily ensured. To make sure that we solved for a pure rotation, find the determinant of $\mathbf{X}$. If $|\mathbf{X}| = 1$ then $\mathbf{R} = \mathbf{X}$. If $|\mathbf{X}| = -1$, $\mathbf{X}$ is a reflection and the desired rotation is

$$\mathbf{X}^* = \mathbf{V}^* \, \mathbf{U}^T.$$

Here $\mathbf{V}^*$ is simply $\mathbf{V}$ with the sign of the third column changed. The desired solution is $\mathbf{R} = \mathbf{X}^*$.

### 5.3 Statistics of least-squares estimate

The analysis in this section is similar to that by Kanatani [9], but more general since we do not have the requirements that $L = 3$ and that the remote receiver directions, $\mathbf{t}'_l$, be orthogonal. The transformation from the true rotation $\mathbf{R}$ to the estimated rotation $\hat{\mathbf{R}}$ is itself a rotation. This error rotation can be specified by an axis unit vector $\mathbf{l}$, and an angle of rotation about that axis, $\Omega$. Letting $\Delta\mathbf{l} = \Omega\,\mathbf{l}$ we have

$$\hat{\mathbf{R}} = \mathbf{R} + \Delta\mathbf{l} \times \mathbf{R} + O(\Delta\mathbf{l})^2, \tag{26}$$

where $O(\Delta\mathbf{l})^2$ is insignificant for small $\Omega$ and the cross-product operates column-wise on $\mathbf{R}$. Define the covariance matrix of the perturbation $\Delta\mathbf{l}$ as

$$V[\hat{\mathbf{R}}] = E\{\Delta\mathbf{l}\Delta\mathbf{l}^T\}. \tag{27}$$

Next, we find $V[\hat{\mathbf{R}}]$ given $E\{\Delta\mathbf{t}_l\Delta\mathbf{t}_l^T\}$, assuming the vectors $\Delta\mathbf{t}_l$ and $\Delta\mathbf{t}_m$ are independent for $l \neq m$.

After substituting into (25) for $\hat{\mathbf{t}}_l$ and $\hat{\mathbf{R}}$ and simplifying,

$$J = \sum_{l=1}^{L} w_l \|\Delta\mathbf{t}_l - (\Delta\mathbf{l} \times \mathbf{R})\mathbf{t}'_l\|^2. \tag{28}$$

Since $\Delta\mathbf{l}$ is chosen to minimize $J$, the gradient of $J$ with respect to $\Delta\mathbf{l}$ must be zero. Applying this rule we find

$$\Delta\mathbf{l} = \mathbf{A}^{-1} \sum_{l=1}^{L} w_l \mathbf{t}_l \times \Delta\mathbf{t}_l, \tag{29}$$

where

$$\mathbf{A} = \sum_{l=1}^{L} w_l (\mathbf{I} - \mathbf{t}_l\mathbf{t}_l^T). \tag{30}$$

We can now form

$$V[\hat{\mathbf{R}}] = \mathbf{A}^{-1} (\sum_{l=1}^{L} w_l^2 [\mathbf{t}_l] E\{\Delta\mathbf{t}_l\Delta\mathbf{t}_l^T\} [\mathbf{t}_l]^T) \mathbf{A}^{-1T}, \tag{31}$$

where $[\mathbf{t}_l]$ is the matrix operator equivalent of the cross-product by $\mathbf{t}_l$, that is, $\mathbf{t}_l \times \mathbf{q} = [\mathbf{t}_l]\mathbf{q}$ for any $\mathbf{q}$.

The rotation error covariance matrix $V[\hat{\mathbf{R}}]$ is used to determine the expected error when pointing a beam. Let $\mathbf{p}'$ be a unit vector specifying a beam direction in the reference coordinate system. To compensate for the array attitude, the beam is pointed in direction $\hat{\mathbf{p}} = \hat{\mathbf{R}}\mathbf{p}'$, but actually goes in direction $\mathbf{p} = \mathbf{R}\mathbf{p}'$. The beam pointing error is

$$\mathbf{p}_e = \hat{\mathbf{p}} - \mathbf{p} = (\Delta\mathbf{l} \times \mathbf{R})\mathbf{p}' = \Delta\mathbf{l} \times \mathbf{p}, \tag{32}$$

and the covariance matrix of this pointing error is

$$\mathbf{C}_p = E\{\mathbf{p}_e\mathbf{p}_e^T\} = [\mathbf{p}]V[\hat{\mathbf{R}}][\mathbf{p}]^T, \tag{33}$$

where $[\mathbf{p}]$ is the matrix operator for cross-product by $\mathbf{p}$.

Using this equation, one can find the expected roll, pitch and yaw errors in the estimate of the array attitude. Suppose roll is rotation about the X-axis; pitch is about the Y-axis; and yaw is about the Z-axis. With $\mathbf{p}' = (0,0,1)^T$, $\mathbf{C}_p(1,1)$ is the variance of $\hat{\mathbf{p}}_e(1)$ and the pitch error. Also, $\mathbf{C}_p(2,2)$ is the variance of $\hat{\mathbf{p}}_e(2)$ and the roll error. Using $\mathbf{p}' = (1,0,0)^T$ yields the pitch and yaw errors and $\mathbf{p}' = (0,1,0)^T$ yields the roll and yaw errors.

### 5.4 Theoretical and experimental performance

Figure 6 illustrates the theoretical and experimental error performance of the least-squares attitude estimation for a situation in which two remote receivers are near the positive Z-axis, in the Y-Z plane and separated by 2.5, 5 or 10 degrees.

To generate the experimental performance data, noisy Tripulse measurements are simulated by adding Gaussian noise to the $T_{xl}$ and $T_{yl}$ values for each remote receiver. A least-squares attitude estimation procedure is performed and the roll, pitch and yaw (as defined above) are found for the attitude error. The rms estimation error using 1000 random trials is plotted versus the noise standard deviation, $\sigma$. The theoretical performance was found by the technique of the previous section, and shown in the plot with dashed curves. As is clear in the plot, it is valid only for small measurement errors.

In the plot, the lowest group of indistinguishable curves are the roll and pitch error for the 2.5, 5 and 10 degree remote receiver separations. Remote receiver separation has no significant effect on roll and pitch error. The yaw error does depend on remote receiver separation with smaller separations causing greater error. The yaw error curves are labeled with the corresponding remote receiver separation angle in the plot.

## 6. Conclusions

This paper presented a novel method for determining the orientation of a remote phased array. The two-step method consists of sensing the actual directions to known terrestrial locations and fusing the direction estimates of two or more participating stations to determine the full 3D pointing information. The underlying monopulse-related method is applicable to receive *or* transmit phased arrays.

The Tripulse method is especially well-suited to active phased array satellites where a high degree of beam pointing stabilization is necessary. Unlike the conventional system where optical sensors drive a physical stabilization subsystem, the orientation estimates may be used to dynamically steer customer beams. As a result, a less accurate but cheaper/lighter suite of sensors may be used to stabilize the satellite bus. Note that such a system is particularly attractive in the case where multiple arrays exist on independently gimbaled panels. Here the Tripulse method is then applied to each panel as opposed to the conventional technique where multiple sets of sensors may be required.

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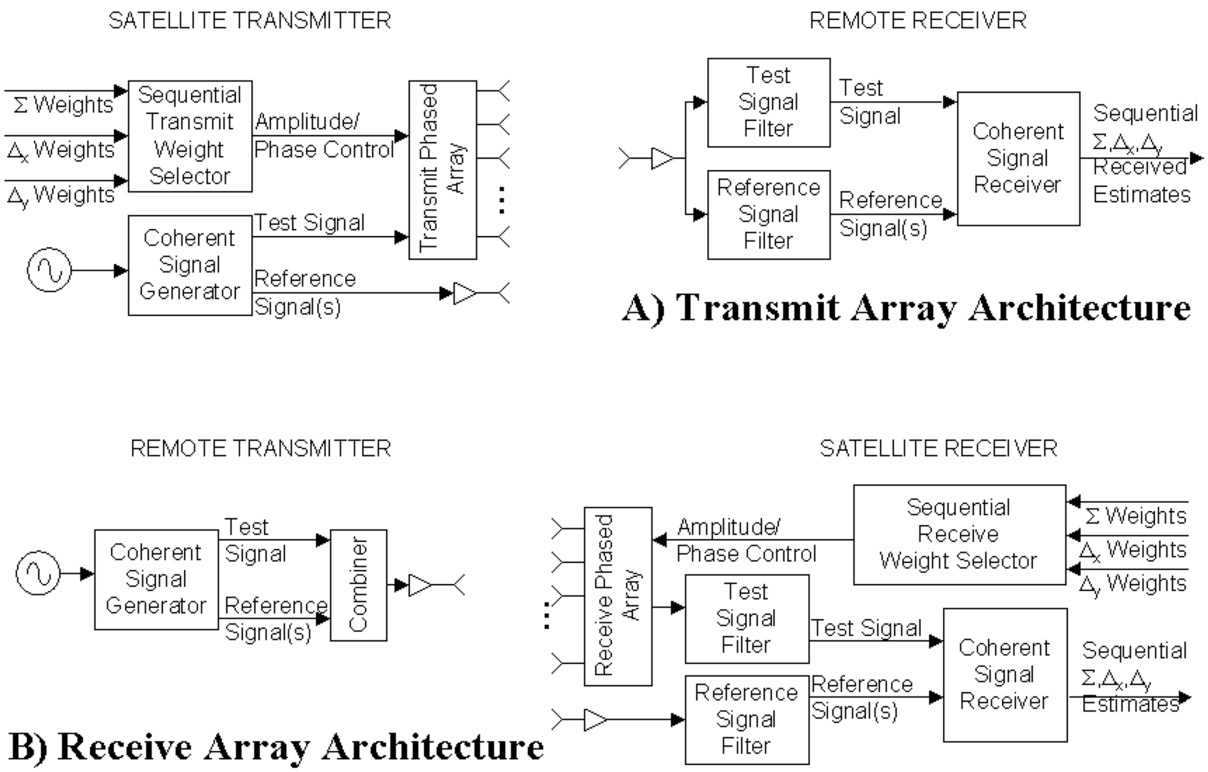


Figure 1: Tripulse Architectures for Transmit/Receive Phased Arrays

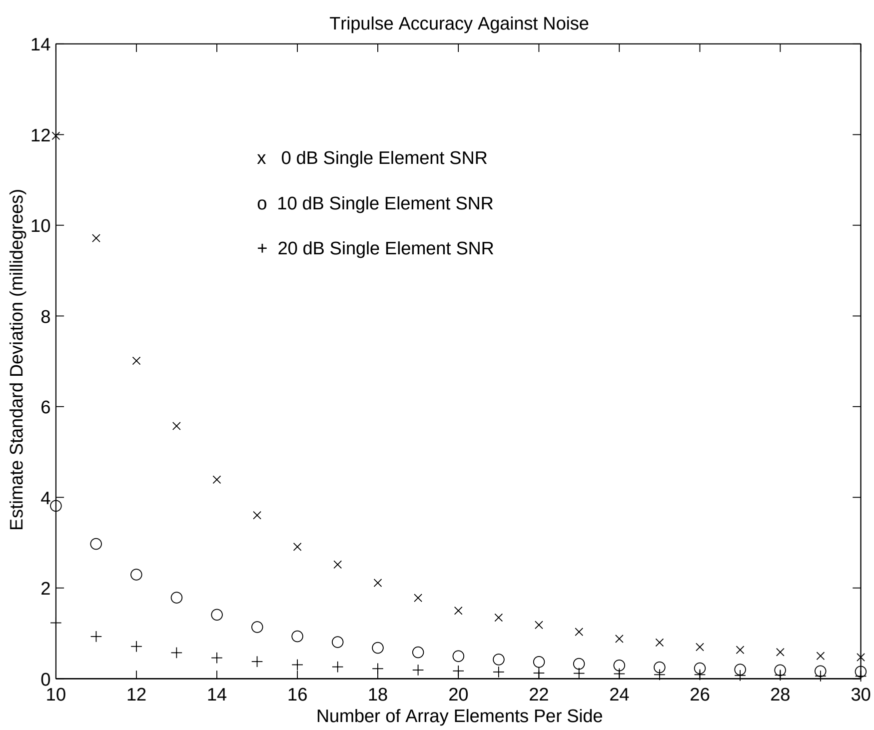


Figure 2: Tripulse robustness to noise

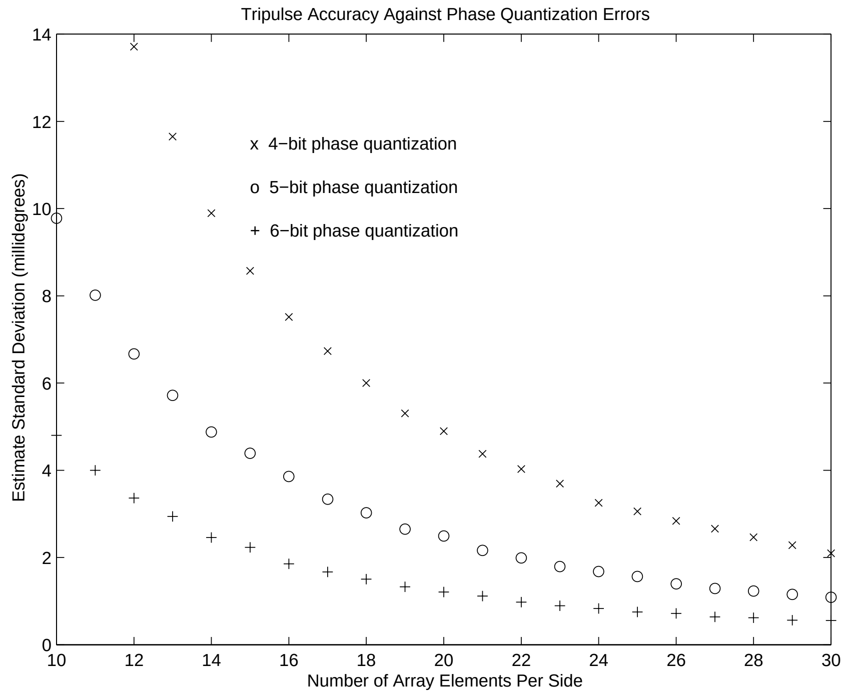


Figure 3: Tripulse robustness to phase quantization errors

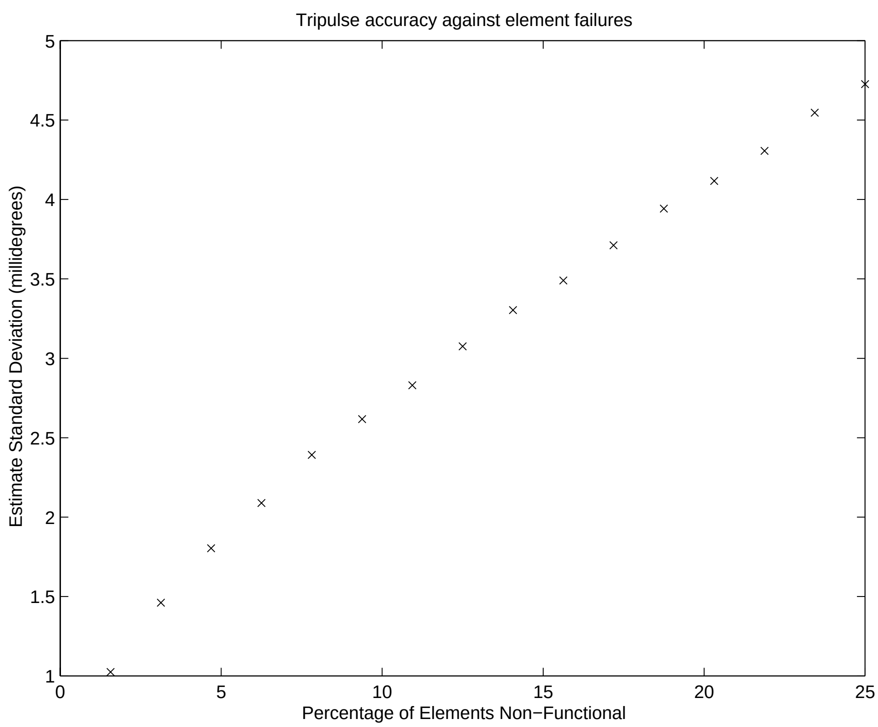


Figure 4: Tripulse robustness to failed elements

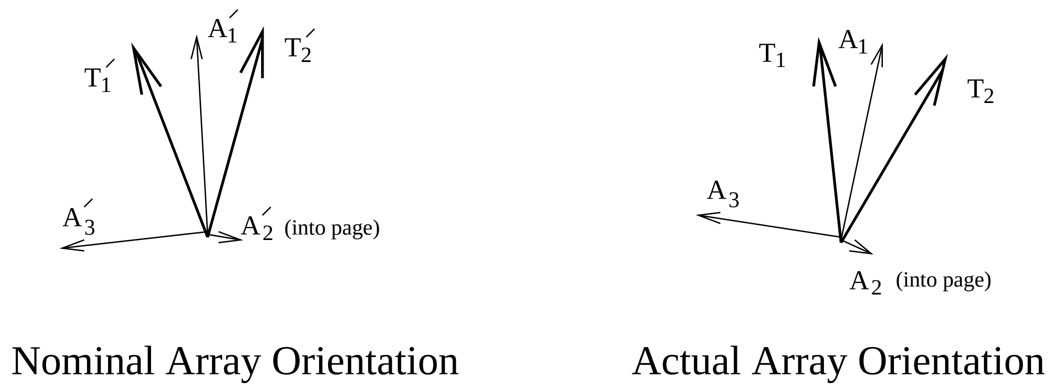


Figure 5: The orthonormal vectors for the nominal array orientation and actual array orientation.

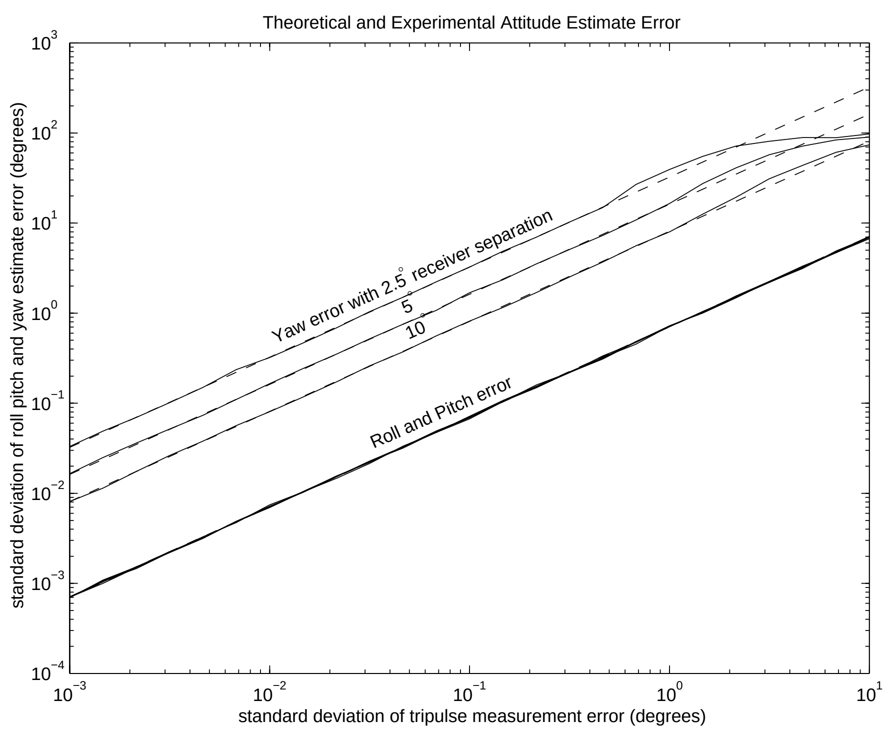


Figure 6: Roll pitch and yaw error versus Tripulse measurement error.